1.1.1 Linearization via Taylor Series

In order to linearize general nonlinear systems, we will use the **Taylor Series** expansion of functions. Consider a function $f(x)$ of a single variable x, and suppose that $\bar{x}$ is a point such that $f(\bar{x}) = 0$. In this case, the point $\bar{x}$ is called an **equilibrium point** of the system $\dot{x} = f(x)$, since we have $\dot{x} = 0$ when $x = \bar{x}$ (i.e., the system reaches an equilibrium at $\bar{x}$). Recall that the Taylor Series expansion of $f(x)$ around the point $\bar{x}$ is given by

$$f(x) = f(\bar{x}) + \left.\frac{df}{dx}\right|_{x=\bar{x}} (x - \bar{x}) + \frac{1}{2} \left.\frac{d^2 f}{dx^2}\right|_{x=\bar{x}} (x - \bar{x})^2 + \frac{1}{6} \left.\frac{d^3 f}{dx^3}\right|_{x=\bar{x}} (x - \bar{x})^3 + \cdots \ .$$

This can be written as

$$f(x) = f(\bar{x}) + \underbrace{\left.\frac{df}{dx}\right|_{x=\bar{x}}}_{a} (x - \bar{x}) + \text{ higher order terms.}$$

For x sufficiently close to $\bar{x}$, these higher order terms will be very close to zero, and so we can drop them to obtain the approximation

$$f(x) \approx f(\bar{x}) + a(x - \bar{x}) \ .$$

Since $f(\bar{x}) = 0$, the nonlinear differential equation $\dot{x} = f(x)$ can be approximated near the equilibrium point by

$$\dot{x} = a(x - \bar{x}) \ .$$

To complete the linearization, we define the **perturbation state** (also known as **delta state**) $\delta x = x - \bar{x}$, and using the fact that $\delta\dot{x} = \dot{x}$, we obtain the linearized model

$$\delta\dot{x} = a\delta x \ .$$

Note that this linear model is valid **only** near the equilibrium point (how "near" depends on how nonlinear the function is).

Extension To Functions of Multiple States and Inputs

The extension to functions of multiple states and inputs is very similar to the above procedure. Suppose the evolution of state x_i is given by

$$\dot{x}_i = f_i(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \ ,$$

for some general function f_i. Suppose that the equilibrium points are given by $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m$, so that

$$f_i(\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n, \bar{u}_1, \bar{u}_2, \ldots, \bar{u}_m) = 0 \quad \forall i \in \{1, 2, \ldots, n\} \ .$$

Note that the equilibrium point should make *all* of the functions f_i equal to zero, so that all states in the system stop moving when they reach equilibrium. The linearization of f_i about the equilibrium point is then given by

$$f_i(x_1, \ldots, x_n, u_1, \ldots, u_m) \approx \sum_{j=1}^{n} \left.\frac{\partial f_i}{\partial x_j}\right|_{x_j=\bar{x}_j} (x_j - \bar{x}_j) + \sum_{j=1}^{m} \left.\frac{\partial f_i}{\partial u_j}\right|_{u_j=\bar{u}_j} (u_j - \bar{u}_j) \ .$$

If we define the delta states and inputs $\delta x_j = x_j - \bar{x}_j$ (for $1 \le j \le n$) and $\delta u_j = u_j - \bar{u}_j$ (for $1 \le j \le m$), the linearized dynamics of state x_i are given by

$$\delta\dot{x}_i = \sum_{j=1}^{n} \left.\frac{\partial f_i}{\partial x_j}\right|_{x_j=\bar{x}_j} \delta x_j + \sum_{j=1}^{m} \left.\frac{\partial f_i}{\partial u_j}\right|_{u_j=\bar{u}_j} \delta u_j \ .$$

Note: Sometimes the "δ" notation is dropped in the linearized equation, with the implicit understanding that we are working with a linearized system.

Example. Linearize the nonlinear state-space model

$$\begin{aligned} \dot{x}_1 &= x_1^2 + \sin x_2 - 1 \\ \dot{x}_2 &= -x_2^3 + u \\ y &= x_1 + x_2 \end{aligned}$$

around the equilibrium point $\bar{x}_1 = 1, \bar{x}_2 = 0, \bar{u} = 0$.
Solution.

2 Obtaining The Transfer Function From A Linear State-Space Model

Since we can generally convert nonlinear models to a linear model (in a small region around the equilibrium point), we will focus on linear state-space models of the form

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}, \quad \mathbf{x} \in \mathbb{R}^n, \mathbf{u} \in \mathbb{R}^m, \mathbf{y} \in \mathbb{R}^p \\ \mathbf{y} &= \mathbf{C}\mathbf{x} \ .\end{aligned}$$

for the rest of the course. Since this model represents a linear system, we can ask how the matrices $\mathbf{A}, \mathbf{B}$ and $\mathbf{C}$ relate to the transfer function of the system. To see this, take the Laplace Transform of the above state space equations:

$$\begin{aligned}s\mathbf{X}(s) - \mathbf{x}(0) &= \mathbf{A}\mathbf{X}(s) + \mathbf{B}\mathbf{U}(s) \\ \mathbf{Y}(s) &= \mathbf{C}\mathbf{X}(s) \ .\end{aligned}$$

Note that this includes the initial conditions of all the states. The first equation can be rearranged to solve for $X(s)$ as follows:

$$(s\mathbf{I} - \mathbf{A})\mathbf{X}(s) = \mathbf{x}(0) + \mathbf{B}\mathbf{U}(s) \Leftrightarrow \mathbf{X}(s) = (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \ .$$

The term $\mathbf{I}$ represents the $n \times n$ identity matrix. Substituting this into the equation for $\mathbf{Y}(s)$, we obtain

$$\mathbf{Y}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{x}(0) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}\mathbf{U}(s) \ .$$

> The transfer function of the state-space model $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$, $\mathbf{y} = \mathbf{C}\mathbf{x}$ (when $\mathbf{x}(0) = 0$) is
>
> $$\mathbf{H}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} \ .$$

Note that $H(s)$ is a $p \times m$ matrix, and thus it is a generalization of the transfer function for standard single-input single-output systems. In fact, it is a matrix where entry i, j is a transfer function describing how the j–th input affects the i–th output. When $p = 1$ and $m = 1$, we get the transfer function that we studied in the first part of the course.

Example. Calculate the transfer function for the state space model

$$\dot{\mathbf{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{\mathbf{A}} \mathbf{x} + \underbrace{\begin{bmatrix} 0 \\ 4 \end{bmatrix}}_{\mathbf{B}} u, \quad y = \underbrace{\begin{bmatrix} 1 & 0 \end{bmatrix}}_{\mathbf{C}} \mathbf{x} \ .$$

Solution.

Note that the above solution agrees with the transfer function at the beginning of the section.